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Hyperbolicity in a double-well potential

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Abstract. We investigate the dynamics of a particle in a double-well potential. There is a critical energy greater than zero beyond which no localized motion is possible. For certain energies smaller than the critical energy, the system can be shown to have a nonempty, bounded invariant set. Conditions are found under which this invariant set is hyperbolic.

1. Introduction

For the dynamical systems theory the concept of hyperbolicity is of central importance (see for example [1]). There are only a few systems for which this property can be proven rigorously. In this note we consider a Hamiltonian scattering system with two degrees of freedom. So far most model investigations have been concerned with arrangements of hard disks or with potentials which are composed of repulsive parts [2, 3]. However, for a large variety of physical scattering problems attractive potential regions play a dominant role. We introduce a Hamiltonian system whose potential consists of two wells. This system can be described by a map. For certain parameter values the map has a nonempty, bounded invariant set. The investigation of this set is essential [4] to understand the scattering behaviour of the system. In this note we derive a criterion for which the invariant set is hyperbolic. A main tool is the construction of invariant sector bundles [5].

A concrete example is given, for which the dynamics on the invariant set is conjugated to a full shift of finite-type.

2. The model

We investigate the motion of a particle of mass, m, in a model-potential which is given by two radial potentials, $v(r)$, with centres on the x-axis at $\boldsymbol{R}_+ = \boldsymbol{R}_0\boldsymbol{e}_x$ and $\boldsymbol{R}_- = -\boldsymbol{R}_0\boldsymbol{e}_x$, $R_0 > 0$. The range of the potential, v, is finite, i.e. outside of this range the action of the potential is negligible and the dynamics is that of a free particle. We measure lengths in units of this range. Furthermore, we only consider the case that the ranges of the potentials, v, do not overlap, i.e. $R_0 > 1$ (see figure 1). For a fixed energy, $E > 0$, the action of the potential $v(r)$ is completely specified by the deflection function $\Theta(l)$ [6]:

$$\Theta(l) = \pi - 2l \int_{\bar{r}}^{\infty} \frac{\mathrm{d}r}{r^2\sqrt{2m(E - v(r)) - \frac{l^2}{r^2}}} \tag{1}$$

where l is the angular momentum of the particle and $\bar{r}$ is the classical outer turning point, i.e. the largest zero of the denominator. Note that time-reversal symmetry implies

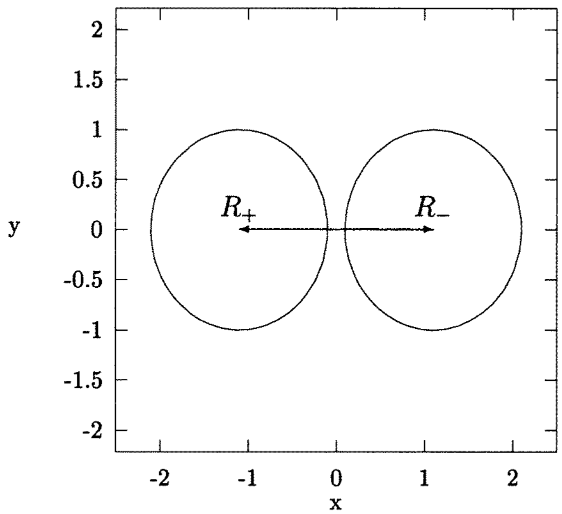

Figure 1. A sketch of the geometry of the system. The interior of the circles corresponds to the regions of nontrivial interactions.

$\Theta(l) = -\Theta(-l)$. Here we are interested in attracting potential wells, i.e. the deflection function is nonpositive. We restrict the discussion to the energy range beyond the orbiting threshold. Furthermore, we assume that for the finite l-interval corresponding to nontrivial scattering trajectories the deflection function is continuously differentiable. We only consider potentials, v, for which the deflection function has a unique minimum. (This minimum is called the rainbow angle (for examples see [6, 7])).

We set $m = 1$ and scale the time according to

$$t \longmapsto \frac{1}{\sqrt{2E}} t. \tag{2}$$

Outside the range of v one therefore obtains for the norm of the momentum

$$|\boldsymbol{p}| = 1. \tag{3}$$

Note that the angular momentum is equal to the impact parameter and that in view of (2) changing the energy is equivalent to changing the unit of time.

We now discuss, how the well system can be described by a map. Suppose that the particle leaves potential i, $i \in \{+, -\}$, with momentum $\boldsymbol{p}$. We denote the angle between $\boldsymbol{p}$ and $\boldsymbol{R}_i$ by β_i and the angular momentum with respect to $\boldsymbol{R}_i$ by l_i. If the particle visits potential j, $j \neq i$, the angle after leaving potential j is given by

$$\beta_j = \beta_i - \pi - \Theta(l_j). \tag{4a}$$

For the angular momentum one easily obtains

$$l_j = l_i + 2R_0 \sin\beta_i. \tag{4b}$$

A necessary condition for visiting potential j is given by

$$|l_j| \leqslant 1. \tag{4c}$$

Furthermore, it is required that the radial momentum with respect to R_j is negative, i.e.

$$\sqrt{1 - l_i^2} + 2\cos\beta_i < 0. \tag{4d}$$

Conditions (4*c*) and (4*d*) combined are also sufficient for visiting potential j.

A triple (β_i, l_i, i) uniquely defines a trajectory of the system. On the other hand, every trajectory which visits at least one well gives rise to a (β_i, l_i, i)-triple. Therefore all nontrivial trajectories of the well system will be taken into account, if one restricts the attention to the system (4). Finite series of (β_i, l_i, i)-triples correspond to trajectories leaving the interaction region both in the past and in the future. Bi-infinite series belong to trajectories staying in the interaction region for all times. Finally, forward-infinite (backward-infinite) series describe trajectories which stay for all positive (negative) times but leave the wells for negative (positive) times.

The last components of the triples yield an alternating series of $+$'s and $-$'s. Thus, the only interesting components are the first and the second ones. We can describe the system by the two-dimensional map

$$F: \quad \beta' = \beta - \pi - \Theta(l') \tag{5a}$$

$$l' = l + 2R_0 \sin\beta. \tag{5b}$$

This map has to be iterated until one of the conditions

$$|l| \leqslant 1 \tag{5c}$$

$$\sqrt{1-l^2} + 2\cos\beta < 0 \tag{5d}$$

is violated. This violation corresponds to the particle escaping from the interaction region.

Formally, the image of F is $S^1 \times \mathbb{R}$. Condition (5*c*) enforces the restriction to the l-interval $[-1, 1]$. Therefore the phase space of the system is the cylinder

$$\Gamma = S^1 \times [-1, 1]. \tag{6}$$

Trajectories remaining in a vicinity of the two wells for all positive and negative times correspond to points in Γ which stay in Γ under F^n for all $n \in \mathbb{Z}$. In the following we are interested in the set, Λ, of these points.

3. A criterion for hyperbolicity

In this section we derive a criterion for F being hyperbolic. First we give a condition for which Λ is not empty.

Theorem 1. The set, Λ, is not empty if and only if the minimum of the deflection function is smaller than or equal to $-\pi$.

In order to prove the theorem it is useful to introduce two transformations. The first is trivial and serves for simplifying formulae. We define the angle-variable

$$\alpha = \beta + \pi. \tag{7}$$

The map F now reads

$$F: \quad \begin{aligned} \alpha' &= \alpha - \pi - \Theta(l') \\ l' &= l - 2R_0 \sin\alpha. \end{aligned} \tag{8}$$

Because of the geometry of the system all points in Λ fulfill the condition

$$-\pi/2 < \alpha < \pi/2. \tag{9}$$

The second transformation is given by

$$T: \quad \begin{aligned} x &= l \\ y &= l - 2R_0 \sin\alpha. \end{aligned} \tag{10}$$

All points of Λ are mapped onto the square $Q = \{(x, y) | -1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1\}$ by T. For two consecutive points (α, l) and (α', l') of a trajectory in Λ, the x-component corresponds to the old angular momentum and the y-component to the new angular momentum. In this sense the transformation T combines two points of a trajectory in Λ to one point in Q. Because of $|x - y| \leqslant 2 < 2R_0$ the map

$$T^{-1}: \quad \begin{aligned} \alpha &= \arcsin\left(\frac{x-y}{2R_0}\right) \\ l &= x \end{aligned} \tag{11}$$

is defined on Q and maps Q onto Γ. Restricted to Λ and $T(\Lambda)$, the maps T and T^{-1} are each other's inverse. T transforms the map F into

$$G: \quad x' = y \tag{12a}$$

$$y' = y - 2R_0 \sin\left(\arcsin\left(\frac{x-y}{2R_0}\right) - \pi - \Theta(y)\right). \tag{12b}$$

Consider a point in Q whose image point (x, y) under G^n stays in Q *and* satisfies the condition

$$\alpha'(x, y) = \arcsin\left(\frac{x-y}{2R_0}\right) - \pi - \Theta(y) \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\tag{13}$$

for all $n \in \mathbb{Z}$. We denote the set of all these points by Λ_G. It is easy to show by direct computation that the following diagram commutes:

$$\begin{array}{ccc} \Lambda & \xrightarrow{F} & \Lambda \\ {\scriptstyle T}\downarrow & & \uparrow{\scriptstyle T^{-1}} \\ \Lambda_G & \xrightarrow{G} & \Lambda_G \end{array} \tag{14}$$

We now prove theorem 1.

Proof of theorem 1. Direct computation shows that an l-value with $\Theta(l) = \pi$ implies a fixed point of F. By the intermediate value theorem such a value of l exists, if the minimum of the deflection function is smaller than or equal to $-\pi$. Therefore, the stated condition is sufficient for nonempty Λ. To see that it is also necessary for us to look at the map G. We show that for $\Theta(l) > -\pi$ the set Λ_G is empty. Since G is invariant under the transformation

$$(x, y) \longmapsto (-x, -y) \tag{15}$$

it is sufficient to verify that every point of

$$Q_0 = \{(x, y) \in Q | y \geqslant -x\} \tag{16}$$

leaves Q under some iterate of G.

Assume $Q_0 \cap \Lambda_G \neq \emptyset$. We divide Q_0 into two parts

$$Q_1 = \{(x, y) \in Q_0 | y \geqslant 0, y \geqslant x\} \tag{17}$$

$$Q_2 = \{(x, y) \in Q_0 | x \geqslant 0, y \leqslant x\}. \tag{18}$$

We show that both $Q_1 \cap \Lambda_G$ and $Q_2 \cap \Lambda_G$ are empty, thus contradicting the assumption $Q_0 \cap \Lambda_G \neq \emptyset$.

First we consider Q_1 and claim that

(1)

$$G(Q_1 \cap \Lambda_G) \subset Q_1 \cap \Lambda_G. \tag{19}$$

(2) There is an $\varepsilon > 0$ such that for all $(x, y) \in Q_1 \cap \Lambda_G$:

$$y' > y + \varepsilon. \tag{20}$$

Note that this implies $Q_1 \cap \Lambda_G = \emptyset$, because for all $(x, y) \in Q_1 \cap \Lambda_G$ all images of (x, y) are also in $Q_1 \cap \Lambda_G$, according to (19). There is an $n > 0$ such that $y + n\varepsilon > 1$ and by (20) $G^n(x, y) \notin Q$. That is a contradiction to $(x, y) \in \Lambda_G$.

So we have to prove relations (19) and (20) in order to show that $Q_1 \cap \Lambda_G$ is empty. Inequality (20) implies (19): by (20) and (17) of Q_1 one obtains for a point $(x, y) \in Q_1 \cap \Lambda_G$:

$$y' > y + \varepsilon \geqslant 0 \tag{21}$$

furthermore with (12) the inequality

$$y' > y + \varepsilon \geqslant y = x' \tag{22}$$

holds. Therefore, $Q_1 \cap \Lambda_G$ is invariant under G.

It remains to prove (20). For a point $(x, y) \in Q_1 \cap \Lambda_G$ one obtains $x - y \leqslant 0$. This implies

$$\arcsin\left(\frac{x-y}{2R_0}\right) \in \left[-\frac{\pi}{2}, 0\right]. \tag{23}$$

Since $y \geqslant 0$ it follows that $\Theta(y) \leqslant 0$. Denoting the minimum of the deflection function by Θ_R one obtains

$$0 \geqslant \Theta(y) \geqslant \Theta_R > -\pi. \tag{24}$$

Assuming $\Theta(y) \in [-\pi/2, 0]$ we conclude

$$\alpha'(x, y) = \arcsin\left(\frac{x-y}{2R_0}\right) - \pi - \Theta(y) \bmod 2\pi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right[\tag{25}$$

and condition (13) is violated. So $\Theta(y)$ lies in the interval $]\Theta_R, -\pi/2[$. It follows that $-\pi - \Theta(y)$ as well as $\alpha'(x, y)$ lie in the interval $[-\pi/2, \pi/2]$. In this interval the sine function is monotonically increasing, such that with (23) we obtain the inequality

$$\sin\alpha' = \sin\left(\arcsin\left(\frac{x-y}{2R_0}\right) - \pi - \Theta(y)\right) \leqslant \sin(-\pi - \Theta(y)). \tag{26}$$

Since $\Theta_R > -\pi$ there is an $\varepsilon > 0$ with $-2R_0 \sin\Theta_R > \varepsilon$. With (12) inequality (20) follows:

$$\begin{aligned} y' &= y - 2R_0 \sin(\alpha') \\ &\geqslant y - 2R_0 \sin(-\pi - \Theta(y)) \\ &\geqslant y - 2R_0 \sin\Theta_R \\ &> y + \varepsilon. \end{aligned} \tag{27}$$

Having proven that $Q_1 \cap \Lambda_G$ is empty, we now want to show the same for $Q_2 \cap \Lambda_G$. We consider the inverse of G:

$$G^{-1}: \quad x' = x - 2R_0 \sin\left(\arcsin\left(\frac{y-x}{2R_0}\right) - \pi - \Theta(x)\right) \tag{28a}$$

$$y' = x. \tag{28b}$$

Note that G^{-1} can formally be obtained from G by interchanging x and y, and x' and y', respectively. The argument of the sine function in (28*a*) is given by

$$\arcsin\left(\frac{y-x}{2R_0}\right) - \pi - \Theta(x) = -\arcsin\left(\frac{x'-y'}{2R_0}\right). \tag{29}$$

Since this argument lies in the interval $]-\pi/2, \pi/2[$ one can prove in the same way as above:

(1)

$$G^{-1}(Q_2 \cap \Lambda_G) \subset Q_2 \cap \Lambda_G. \tag{30}$$

(2) There is an $\varepsilon > 0$ such that for all $(x, y) \in Q_2 \cap \Lambda_G$:

$$x' > x + \varepsilon. \tag{31}$$

This implies that $Q_2 \cap \Lambda_G$ is empty and we conclude: $\Lambda_G = \emptyset$, which proves theorem 1. □

We now give a criterion for hyperbolicity.

Theorem 2. Λ has a hyperbolic structure if

$$\min_{(\beta,l)\in\Lambda} (R_0|\Theta'(l)|) > \max_{(\beta,l)\in\Lambda} \left(\frac{-2}{\cos\beta}\right). \tag{32}$$

To prove this theorem we show that there are invariant sector bundles [5, 9]. We need the following lemma.

Lemma. If there is an $0 < \epsilon < 1$ such that for all $(x, y) \in \Lambda_G$ the inequality

$$\frac{1 + \left|2R_0 \cos\alpha'(x, y) \frac{\partial\alpha'(x,y)}{\partial x}\right|}{\left|1 - 2R_0 \cos\alpha'(x, y) \frac{\partial\alpha'(x,y)}{\partial y}\right|} < \epsilon \tag{33}$$

holds, then Λ_G has a hyperbolic structure.

Proof of the lemma. We will show that the constant sector bundles

$$S_\epsilon^u = \{(\xi, \eta) | |\xi| \leqslant \epsilon|\eta|\} \tag{34a}$$

$$S_\epsilon^s = \{(\xi, \eta) | |\eta| \leqslant \epsilon|\xi|\} \tag{34b}$$

are invariant under the Jacobians DG and DG^{-1}, respectively. With

$$\alpha'_x(x, y) = \frac{\partial\alpha'(x, y)}{\partial x} = \frac{1}{2R_0\sqrt{1 - \left(\frac{x-y}{2R_0}\right)^2}} \tag{35}$$

and

$$\alpha'_y(x, y) = \frac{\partial\alpha'(x, y)}{\partial y} = \frac{-1}{2R_0\sqrt{1 - \left(\frac{x-y}{2R_0}\right)^2}} - \Theta'(y) \tag{36}$$

one obtains

$$DG|_{(x,y)} = \begin{pmatrix} 0 & 1 \\ -2R_0 \cos(\alpha'(x, y))\alpha'_x(x, y) & 1 - 2R_0 \cos(\alpha'(x, y))\alpha'_y(x, y) \end{pmatrix}. \tag{37}$$

To simplify formulae we introduce the abbreviations

$$H_1(x, y) = 1 - 2R_0 \cos(\alpha'(x, y))\alpha'_y(x, y) \tag{38}$$

$$H_2(x, y) = -2R_0 \cos(\alpha'(x, y))\alpha'_x(x, y). \tag{39}$$

The hypothesis in the lemma now is given by

$$\frac{1 + |H_2|}{|H_1|} < \epsilon. \tag{40}$$

Suppose $(\xi_0, \eta_0) \in S^u_\epsilon$ and $(\xi_1, \eta_1) = DG(\xi_0, \eta_0)$. At first we show that the η-component will be elongated under DG by at least a factor of $1/\epsilon$:

$$\begin{aligned} |\eta_1| &= |H_2\xi_0 + H_1\eta_0| \\ &\geqslant |H_1||\eta_0| - |H_2||\xi_0|. \end{aligned} \tag{41}$$

Since by definition (34*a*) $|\eta_0| \geqslant |\xi_0|/\epsilon$ one obtains

$$|\eta_1| \geqslant \left(|H_1| - \frac{1}{\epsilon}|H_2|\right)|\eta_0|. \tag{42}$$

With (40) the inequality

$$|\eta_1| > \frac{1}{\epsilon}|\eta_0| \tag{43}$$

follows.

By (37) $\xi_1 = \eta_0$, and therefore

$$|\eta_1| > \frac{1}{\epsilon}|\xi_1| \tag{44}$$

i.e. $(\xi_1, \eta_1) \in S^u_\epsilon$. This proves the invariance of the bundle (34*a*) under DG.

To obtain an estimate of the norm of (ξ_1, η_1) we use (43) and $(\xi_0, \eta_0) \in S^u_\epsilon$ (i.e. $|\xi_0| \leqslant \epsilon|\eta_0| = \epsilon|\xi_1|$) to conclude

$$|(\xi_1, \eta_1)| \geqslant \frac{1}{\epsilon}|(\xi_0, \eta_0)|. \tag{45}$$

Thus, we have shown that S^u_ϵ is invariant under DG and the length of a vector in S^u_ϵ is elongated under DG by at least a factor of $1/\epsilon$.

In order to complete the proof of the lemma, the corresponding result for S^s_ϵ and DG^{-1} has to be shown [5]. Due to symplecticity of the underlying system one may use a result presented in [8]. There it is shown that for symplectic systems only one time direction has to be considered. Here, we choose a direct way which is based on the reversible symmetry of the well system. With

$$J : \quad \begin{aligned} x &= y \\ y &= x \end{aligned} \tag{46}$$

we obtain $G^{-1} = JGJ$. The relation $DJS^s_\epsilon = S^u_\epsilon$ implies: $DG^{-1}S^s_\epsilon \subset S^s_\epsilon$. Since the norm is invariant under DJ one immediately obtains the desired result and the lemma is proven. □

Proof of theorem 2. To prove theorem 2 we express the condition of the lemma in (β, l)-coordinates. Using (35), (36) and (11) we obtain:

$$\alpha'_x(\alpha, l) = \frac{1}{2R_0|\cos\alpha|} \tag{47}$$

$$\alpha'_y(\alpha, l) = \frac{-1}{2R_0|\cos\alpha|} - \Theta'(l'). \tag{48}$$

The left-hand side of (33) is given by:

$$\frac{1 + \left|2R_0\cos\alpha' \frac{\partial\alpha'(x,y)}{\partial x}\right|}{\left|1 - 2R_0\cos\alpha'(x,y)\frac{\partial\alpha'(x,y)}{\partial y}\right|} = \frac{1 + \left|\frac{\cos\alpha'}{\cos\alpha}\right|}{\left|1 + \frac{\cos\alpha'}{|\cos\alpha|} + 2R_0\cos\alpha'\Theta'(l')\right|}. \tag{49}$$

Note that α and α' are contained in the interval $]-\pi/2, \pi/2[$. Therefore, both $\cos\alpha$ and $\cos\alpha'$ are larger than zero. By the lemma the invariant set has a hyperbolic structure if

$$1+\frac{\cos\alpha'}{\cos\alpha} < \epsilon\left|1+\frac{\cos\alpha'}{\cos\alpha}+2R_0\Theta'(l')\cos\alpha'\right|. \tag{50}$$

For $\Theta'(l') > 0$ this inequality holds if and only if

$$1+\frac{\cos\alpha'}{\cos\alpha} < \epsilon\left(1+\frac{\cos\alpha'}{\cos\alpha}+2R_0\Theta'(l')\cos\alpha'\right). \tag{51}$$

With (7) this is equivalent to

$$R_0\Theta'(l') > \left(\frac{\epsilon-1}{2\epsilon}\right)\left(\frac{1}{\cos\beta'}+\frac{1}{\cos\beta}\right). \tag{52}$$

For $\Theta'(l') < 0$, (50) is satisfied if and only if

$$1+\frac{\cos\alpha'}{\cos\alpha}+2R_0\Theta'(l')\cos\alpha' < -\frac{1}{\epsilon}\left(1+\frac{\cos\alpha'}{\cos\alpha}\right). \tag{53}$$

This is equivalent to

$$R_0\Theta'(l') < \left(\frac{\epsilon+1}{2\epsilon}\right)\left(\frac{1}{\cos\beta'}+\frac{1}{\cos\beta}\right). \tag{54}$$

Note that both sides of (32) are defined since Λ is compact and the occurring functions are continuous. If (32) is satisfied then there is a $\mu > 1$ with

$$\min_{(\beta,l)\in\Lambda}(R_0|\Theta'(l)|) > \mu \max_{(\beta,l)\in\Lambda}\left(\frac{-2}{\cos\beta}\right). \tag{55}$$

With the definition $\epsilon = 1/(2\mu-1)$ the relation $0 < \epsilon < 1$ holds. Hence, for all $(\beta, l) \in \Lambda$:

$$\begin{aligned} R_0|\Theta'(l')| &\geqslant \min_{(\beta,l)\in\Lambda}(R_0|\Theta'(l)|) \\ &> \mu \max_{(\beta,l)\in\Lambda}\left(\frac{-2}{\cos\beta}\right) \\ &\geqslant -\frac{1+\epsilon}{2\epsilon}\left(\frac{1}{\cos\beta'}+\frac{1}{\cos\beta}\right). \end{aligned} \tag{56}$$

Since $(1-\epsilon) < (1+\epsilon)$, relations (54) and (52) prove the theorem. □

Verifying condition (32) for a given system is much easier than verifying, for example, condition (33). In (32) only projections onto the coordinate axes have to be considered which is a great advantage. To elucidate our results we give an example in the following section.

4. An example

As a deflection function we choose

$$\Theta(l) = \begin{cases} \dfrac{3\sqrt{3}}{2}k(l^3-l)\pi & |l| \leqslant 1 \\ 0 & |l| > 1 \end{cases} \tag{57}$$

with $k > 0$. The rainbow angle is given by $-k\pi$. For this deflection function the inverse scattering problem can be solved. As shown in the appendix, a potential corresponding to (57) is given by

$$(r(\gamma), v(\gamma)) = (\gamma \mathrm{e}^{-\sqrt{3}k(1-\gamma^2)^{\frac{3}{2}}}, \tfrac{1}{2}(1-\mathrm{e}^{2\sqrt{3}k(1-\gamma^2)^{\frac{3}{2}}})) \tag{58}$$

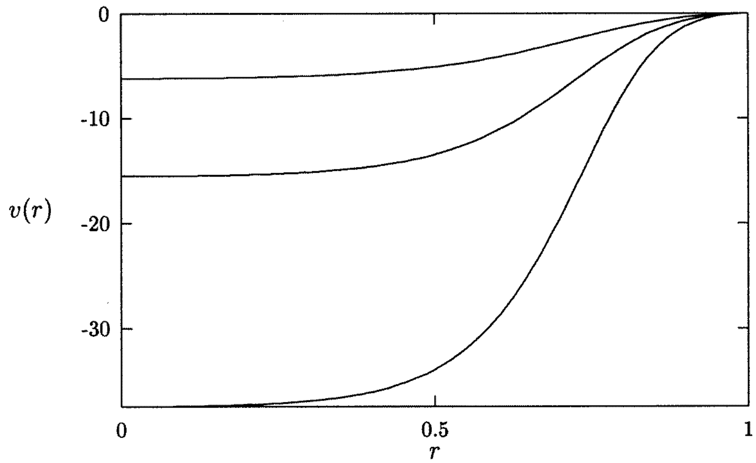

Figure 2. Potential corresponding to the deflection function (57). From top to bottom the potentials $v(r)$ for $k = 0.75, 1.0, 1.25$ are shown.

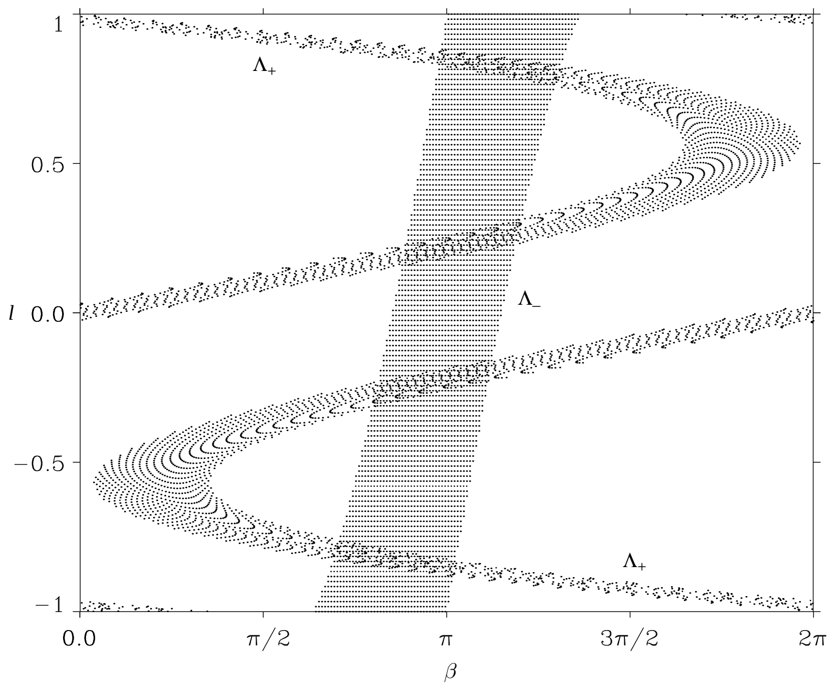

Figure 3. The sets Λ_- and Λ_+ for $R_0 = 1.1$ and $k = 1.9$.

with $\gamma \in [0, 1]$ and $v(r) = 0$ for $r > 1$. Figure 2 shows the potential for several values of k. Note that the potential depth is a monotonically increasing function of k. This means that one can interprete increasing k as decreasing energy.

For $k = 1.9$ and $R_0 = 1.1$ figure 3 shows the set, Λ_-, of points in Γ which have at least one image point under F in Γ, and the set, Λ_+, of points which have at least one preimage point in Γ. Λ is contained in the intersection of these two sets. Figure 3 gives

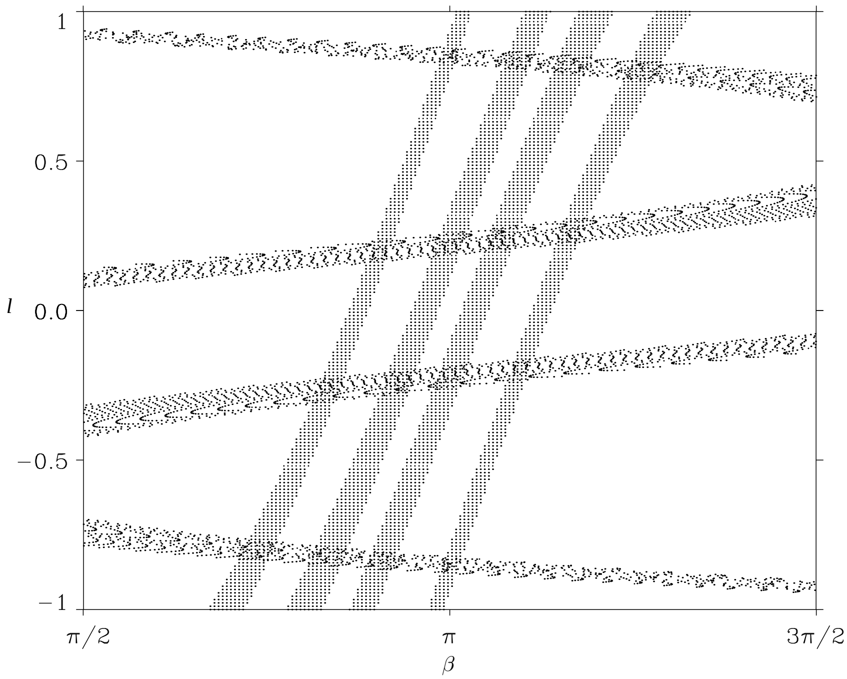

Figure 4. The part $\bar{\Gamma}$ of Γ.

rise to the rough estimate that the absolute values of l-components of points in Λ are either greater than 0.7 or smaller than 0.35. Furthermore, the β-components lie in the interval $[0.6\pi, 1.4\pi]$, such that by inserting the limiting values we obtain the estimate

$$\min_{(\beta,l)\in\Lambda}(R_0|\Theta'(l)|) \geqslant 8.01 > 6.48 \geqslant \max_{(\beta,l)\in\Lambda}\left(\frac{-2}{\cos\beta}\right). \tag{59}$$

Hence, for the chosen parameter values the system is hyperbolic by theorem 2.

We now discuss the structure of Λ. The β-components of points in the invariant set are contained in the interval $[\pi/2, 3\pi/2]$. In figure 4 the part $\bar{\Gamma} = [\pi/2, 3\pi/2] \times [-1, 1]$ of Γ is shown. The four vertical stripes correspond to points whose image points under F are contained in $\bar{\Gamma}$, the four horizontal stripes correspond to points which have at least one pre-image point in $\bar{\Gamma}$. Note that Λ is contained in the intersection of the horizontal and vertical stripes. From left to right the vertical stripes are mapped onto the horizontal stripes from top to bottom. Since the horizontal (vertical) boundaries of the vertical stripes are mapped onto the horizontal (vertical) boundaries of the horizontal stripes, it can be shown by standard arguments, as discussed for example in [9], that Λ is a *Cantor set* and the dynamics on Λ is conjugated to the full shift on a symbol space with four symbols.

5. Concluding remarks

It is easy to generalize the above results to a system consisting of N wells whose centres form a regular N-polygon and the distances of which are large enough such that on any straight line there are at most two wells. In the case of hyperbolicity a complete symbolic dynamics can be introduced. With respect to this symbolic dynamics the model discussed above exhibits a rich behaviour. A detailed discussion of this symbolic dynamics and its parameter dependence is the topic of a forthcoming paper. We close with the remark that by using the technique presented in section 3 it is an easy task to derive a criterion for the hyperbolicity of the model introduced by Troll and Smilansky in [11].

Appendix

In this appendix we solve the inverse scattering problem (57). The method is discussed in [7].

In our scaling the potential is identically zero for $r > 1$. Thus, we obtain from (1):

$$\Theta(l) = \pi - 2\int_{\bar{r}}^{\infty} \frac{\mathrm{d}r}{r^2\sqrt{\frac{(1-2v(r))}{l^2} - \frac{1}{r^2}}} \tag{60}$$

$$= \pi - 2\int_{1}^{\infty} \frac{\mathrm{d}r}{r^2\sqrt{\frac{1}{l^2} - \frac{1}{r^2}}} - 2\int_{\bar{r}}^{1} \frac{\mathrm{d}r}{r^2\sqrt{\frac{(1-2v(r))}{l^2} - \frac{1}{r^2}}} \tag{61}$$

$$= 2\arccos l - 2\int_{\bar{r}}^{1} \frac{\mathrm{d}r}{r^2\sqrt{\frac{(1-2v(r))}{l^2} - \frac{1}{r^2}}}. \tag{62}$$

A main problem in solving the integral is the implicit dependence of the lower integral limit from v. To get rid of this problem we introduce new variables:

$$y = \frac{1}{l^2} \tag{63}$$

$$x = \frac{1}{(1-2v(r))r^2}. \tag{64}$$

Since purely attractive potentials are given by $v(r) \leqslant 0$ for all r and $\mathrm{d}v(r)/\mathrm{d}r > 0$ for $r \in]0,1[$, we have $x \in [1,\infty[$ and $\mathrm{d}x/\mathrm{d}r < 0$ for $r \in]0,1[$. With (63) and (64) by (62) one obtains:

$$\Theta(y) = 2\arccos\frac{1}{\sqrt{y}} - 2\int_{y}^{1} \frac{1}{\sqrt{y-x}} \frac{\sqrt{x}}{r(x)} \frac{\mathrm{d}r(x)}{\mathrm{d}x}\,\mathrm{d}x. \tag{65}$$

With

$$g(x) = -\frac{\sqrt{x}}{r(x)} \frac{\mathrm{d}r(x)}{\mathrm{d}x} \tag{66}$$

and

$$\tilde{\Theta}(y) = -\frac{3\sqrt{3}}{4}k(y^{-\frac{3}{2}} - y^{-\frac{1}{2}})\pi + \arccos\frac{1}{\sqrt{y}} \tag{67}$$

we obtain:

$$\tilde{\Theta}(y) = \int_{1}^{y} \mathrm{d}x\, \frac{g(x)}{\sqrt{y-x}}\,\mathrm{d}x. \tag{68}$$

This is an Abelian integral equation with the solution [10]:

$$g(x) = \frac{1}{\pi}\left(\frac{\tilde{\Theta}(1)}{\sqrt{x-1}} + \int_{1}^{x} \frac{\mathrm{d}\tilde{\Theta}(y)}{\mathrm{d}y} \frac{\mathrm{d}y}{\sqrt{x-y}}\right). \tag{69}$$

A short calculation yields:

$$g(x) = \frac{1}{2\sqrt{x}} + 3\sqrt{3}k\frac{\sqrt{x-1}}{2x^2}. \tag{70}$$

By (64) and (66) the following relation between $g(x)$ and $v(r(x))$ holds:

$$\frac{\mathrm{d}\ln(1-2v(r(x)))}{\mathrm{d}x} = \frac{\mathrm{d}\ln\frac{1}{xr^2(x)}}{\mathrm{d}x} = -\frac{1}{x} + \frac{2g(x)}{\sqrt{x}}. \tag{71}$$

Equation (70) implies:

$$\frac{\mathrm{d}\ln(1-2v(r(x)))}{\mathrm{d}x} = 3\sqrt{3}k\frac{\sqrt{x-1}}{x^{\frac{5}{2}}}. \tag{72}$$

Taking into account that $v(r(x=1)) = 0$ we obtain by integration:

$$v(r(x)) = \tfrac{1}{2}(1 - \mathrm{e}^{2\sqrt{3}k(\sqrt{\frac{x-1}{x}})^3}). \tag{73}$$

From (64) the parametrization of r now immediately follows

$$r(x) = \frac{1}{\sqrt{x}}\mathrm{e}^{-\sqrt{3}k(\sqrt{\frac{x-1}{x}})^3}. \tag{74}$$

The last step is to define $\gamma = 1/\sqrt{x}$.

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